Polynomial Problem - Solution

Part B

We will first answer part B, which will obviously also be a possible solution for part A.
The most important step is to figure out a way to find one root of the polynomial P. This is done using Newton's method, which works as follows. The goal of this method is iteratively try a root of the polynomial, and try to get numerically very close to a root after many iterations. At first, we start with a random possible root, say α_0. Then in each step, we calculate α_{i+1} from α_i in the following way: we approximate P at α_i by a first order Taylor approximation. Then we calculate the root of this approximation, and we take that as α_{i+1}. The mathematical expression of α_{i+1} is thus

$$\alpha_{i+1} = \alpha_i - \frac{P(\alpha_i)}{P'(\alpha_i)},$$

where P' denotes the derivative of P. It can be proven that this iterative process indeed converges at a root of P. Another important remark is that this iterative process doesn't work if $P'(\alpha_i) = 0$, so when α_i is a local extremum for some i. Note that this happens with probability 0, so this shouldn't be a problem. However, one should take this into account when choosing α_0! (i.e., choose α_0 randomly)
The next hard step is to find all other roots. To ensure we don't get the same root again and again using Newton's method, and to ensure we get the right multiplicities for the different roots, we divide the polynomial by the root we just found via Newton's method and iterate this process until we end up with a constant polynomial.
Finally, don't forget to output the found roots in increasing order

Part A alternative method

For a polynomial with only integer coefficients, one could use the following:

Theorem 1. *Any integer root of a polynomial $P(x) = a_n x^n + \ldots + a_0$ is a divisor of the tail coefficient a_0.*

Hence, for part A one could also try to find all divisors of a_0 and then try all of them one by one to find all roots.